Exercise 23

Use the definitions of the hyperbolic functions to find each of the following limits.

(a) $\lim_{x\to\infty} \tanh x$ (b) $\lim_{x\to-\infty} \tanh x$

(c) $\lim_{x\to\infty} \sinh x$ (d) $\lim_{x\to-\infty} \sinh x$

(e) $\lim_{x\to\infty} \operatorname{sech} x$ (f) $\lim_{x\to\infty} \coth x$

(g) $\lim_{x\to 0^+} \coth x$ (h) $\lim_{x\to 0^-} \coth x$

(i) $\lim_{x\to-\infty} \operatorname{csch} x$ (j) $\lim_{x\to\infty} \dfrac{\sinh x}{e^x}$

Solution

Write each of the hyperbolic functions in terms of exponential functions in order to evaluate the limits.

$$\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{\sinh x}{\cosh x} = \lim_{x\to\infty} \frac{\frac{e^x-e^{-x}}{2}}{\frac{e^x+e^{-x}}{2}} = \lim_{x\to\infty} \frac{e^x-e^{-x}}{e^x+e^{-x}} \times \frac{e^{-x}}{e^{-x}} = \lim_{x\to\infty} \frac{1-e^{-2x}}{1+e^{-2x}} = \frac{1-0}{1+0} = 1$$

$$\lim_{x\to-\infty} \tanh x = \lim_{x\to-\infty} \frac{\sinh x}{\cosh x} = \lim_{x\to-\infty} \frac{\frac{e^x-e^{-x}}{2}}{\frac{e^x+e^{-x}}{2}} = \lim_{x\to-\infty} \frac{e^x-e^{-x}}{e^x+e^{-x}} \times \frac{e^{x}}{e^{x}} = \lim_{x\to-\infty} \frac{e^{2x}-1}{e^{2x}+1} = \frac{0-1}{0+1} = -1$$

$$\lim_{x\to\infty} \sinh x = \lim_{x\to\infty} \frac{e^x-e^{-x}}{2} = \frac{\infty-0}{2} = \infty$$

$$\lim_{x\to-\infty} \sinh x = \lim_{x\to-\infty} \frac{e^x-e^{-x}}{2} = \frac{0-\infty}{2} = -\infty$$

$$\lim_{x\to\infty} \operatorname{sech} x = \lim_{x\to\infty} \frac{1}{\cosh x} = \lim_{x\to\infty} \frac{1}{\frac{e^x+e^{-x}}{2}} = \lim_{x\to\infty} \frac{2}{e^x+e^{-x}} = \frac{2}{\infty+0} = 0$$

$$\lim_{x\to\infty} \coth x = \lim_{x\to\infty} \frac{\cosh x}{\sinh x} = \lim_{x\to\infty} \frac{\frac{e^x+e^{-x}}{2}}{\frac{e^x-e^{-x}}{2}} = \lim_{x\to\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}} \times \frac{e^{-x}}{e^{-x}} = \lim_{x\to\infty} \frac{1+e^{-2x}}{1-e^{-2x}} = \frac{1+0}{1-0} = 1$$

$$\lim_{x\to 0^+} \coth x = \lim_{x\to 0^+} \frac{\cosh x}{\sinh x} = \lim_{x\to 0^+} \frac{\frac{e^x+e^{-x}}{2}}{\frac{e^x-e^{-x}}{2}} = \lim_{x\to 0^+} \frac{e^x+e^{-x}}{e^x-e^{-x}} = \frac{e^0+e^0}{e^0-e^0} = \frac{2}{0^+} = \infty$$

$$\lim_{x\to 0^-} \coth x = \lim_{x\to 0^-} \frac{\cosh x}{\sinh x} = \lim_{x\to 0^-} \frac{\frac{e^x+e^{-x}}{2}}{\frac{e^x-e^{-x}}{2}} = \lim_{x\to 0^-} \frac{e^x+e^{-x}}{e^x-e^{-x}} = \frac{e^0+e^0}{e^0-e^0} = \frac{2}{0^-} = -\infty$$

$$\lim_{x\to-\infty} \operatorname{csch} x = \lim_{x\to-\infty} \frac{1}{\sinh x} = \lim_{x\to-\infty} \frac{1}{\frac{e^x-e^{-x}}{2}} = \lim_{x\to-\infty} \frac{2}{e^x-e^{-x}} = \frac{2}{0-\infty} = 0$$

$$\lim_{x\to\infty} \frac{\sinh x}{e^x} = \lim_{x\to\infty} \frac{\frac{e^x - e^{-x}}{2}}{e^x} = \lim_{x\to\infty} \frac{e^x - e^{-x}}{2e^x} = \lim_{x\to\infty} \frac{1 - e^{-2x}}{2} = \frac{1-0}{2} = \frac{1}{2}$$